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LETTER TO THE EDITOR

Further results of enumeration of directed animals on two-dimensional lattices

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Abstract. In a previous paper we gave enumerations and results of the enumeration of directed animals of three different forms (sites, bonds, and bonds without loops) on a variety of two-dimensional lattices. This letter adds results for site animals without loops, and gives an exact solution for the sites (no loops) Kagomé lattice, which is actually a pair of directed self-avoiding walks.

Two-dimensional directed animals are connected clusters, such that each occupied site is either the unique base site, or is one step in a preferred direction from some other site in the animal. An analytic solution for the number of site animals on the square and triangular lattices has been found (e.g. [2]), and proved in many ways (e.g. [3, 4]). In [1], seven other lattices were investigated, as shown in figure 1, where the preferred direction is up and to the right. Three types of animals were studied: site animals, bond animals, and bond animals without loops. The latter are commonly called trees (e.g. [5]). In each case an asymptotic growth in the number of animals of size n of the form $n^{-1/2}\mu^n$ was observed and precise estimates of μ were given for the three animal types on the nine lattices.

This analysis has been repeated for site animals without loops, and the results are given in table 1. A site animal without loops is a directed site animal such that no occupied site is in the preferred direction of more than one occupied site in the animal. For instance, on the square lattice, an animal with four sites arranged in a square is illegal, as the site furthest from the base site can be reached from both the site to the left and the site below.

An initial surprise comes from the fact that the number of site animals without loops on the Kagomé lattice does not behave in the same way as all the other 35 animals studied: the number of animals appears to grow like μ^n rather than the normal $n^{-1/2}\mu^n$. On close inspection of the lattice, the reason for this is clear. The only site that can have more than one site emanating from it is the origin. Site types 1 and 2 (the numbers inside the circles in figure 1) form loops immediately if both sites in the preferred directions are occupied; site types 0 are a little more subtle: one of the possible output directions will be rendered illegal by the type-0 site and its predecessor. Only the origin, with no predecessor is immune to this restriction. Thus the problem reduces to two self-avoiding directed walks on the Kagomé lattice. This structure cannot really be called a two-dimensional animal, so universality is not violated.

It is possible to find the connective constant analytically for this type of graph. It will be identical to the connective constant for one self-avoiding directed walk on the Kagomé

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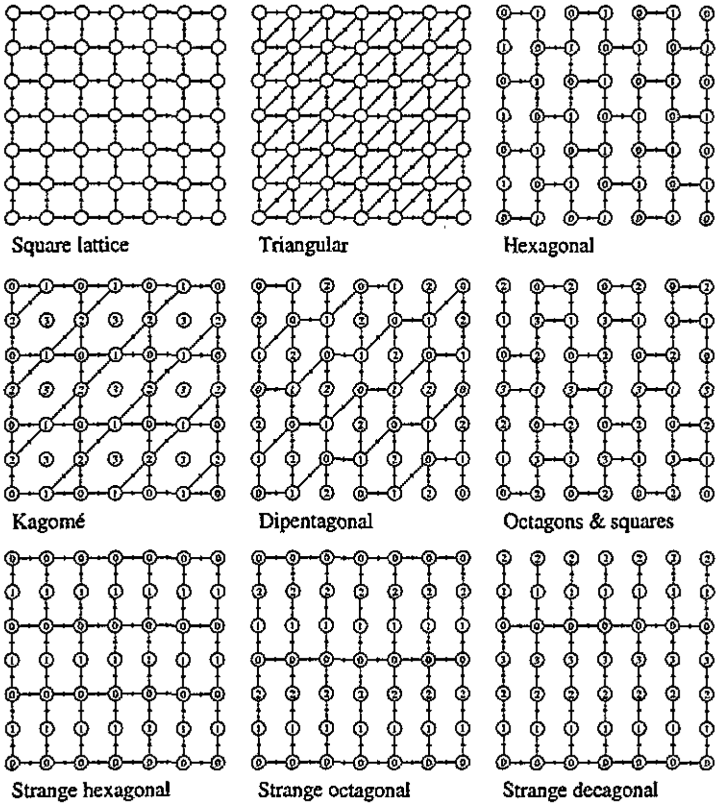

Figure 1. The shapes of the nine lattices studied. The preferred direction is to the upper right.

Table 1. *Summary of results for the nine lattices for site directed animals with no loops.*

Lattice	Terms	Connective constant
Square	40	2.71261 ± 5
Triangular	20	3.0772 ± 5
Hexagonal	70	1.925788 ± 10
Kagomé	31	$\frac{1}{2}(1+\sqrt{5}) = 1.618\ldots$
Dipentagonal	35	2.64160 ± 2
Octagons and squares	70	1.779 ± 1
Strange hexagonal	40	2.526673 ± 4
Strange octagonal	40	2.37813 ± 2
Strange decagonal	30	2.2728 ± 10

lattice. Suppose t_n is the number of single directed walks with n sites not including the origin, and $s_{a,b}$ is the number of self-avoiding walk pairs, where one has a sites and the other has b sites, not including the origin. Then $s_{n,0} = t_n$ and $s_{a,b} \leqslant t_a t_b$. The total number of self-avoiding walk pairs is then $S_n = \sum_{a+b=n} s_{a,b}$, which is bounded $2t_n \leqslant S_n \leqslant (n+1)t_n$. Thus the connective constants are identical.

The value of this connective constant for single directed walks can be established. Define $A(x)$ as the generating function for walks starting from a type-2 site. By symmetry, this will be the same as the generating function for walks starting from a 1 site. Then

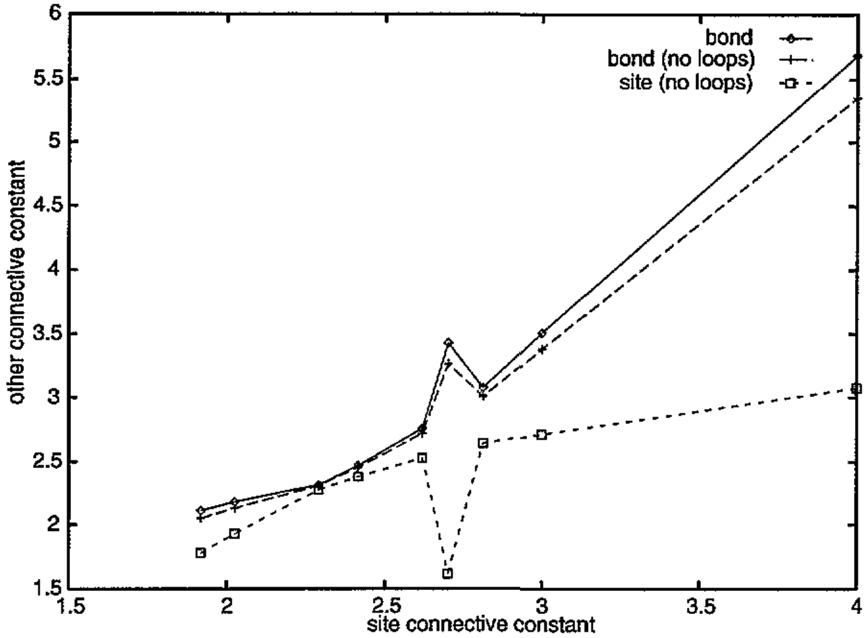

Figure 2. Graph of connective constants of three varieties of directed animals against the connective constant for site animals for the nine lattices in figure 1.

$A = 1 + x + xA + x^2A$, so

$$A = \frac{1+x}{1-x-x^2} = \sum_n t_n x^n .$$

A little algebra gives $t_n \sim \left[\frac{1}{2}(1+\sqrt{5})\right]^n$. Thus the connective constant μ for directed site (no loops) animals on the Kagomé lattice is $\frac{1}{2}(1+\sqrt{5})$. Note that t_n happens to be the $(n+2)$th Fibonacci number.

A longer proof gives an exact generating function for these 'animals' on the Kagomé lattice, and shows an asymptotic behaviour like μ^n. This proof is given in the appendix.

All other results in table 1 are obtained using the method of differential approximants [6] as in [1]. Actual enumerations are not given for reasons of space; they can be obtained from the author.

For ease of visualization, a graph is given in figure 2 plotting the connective constants for bond, bond (no loops) and site (no loops) directed animals against the connective constant for site animals on the same lattice, using the data from [1] table 1. It is very smooth apart from an irregularity for the Kagomé lattice. The latter is expected for the site (no loops) case for reasons mentioned above. The irregularity in the bond animals case is probably due to the site animals' connective constant being artificially low for similar reasons; the triangles are more combinatorially significant for animals based on bonds rather than sites.

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Appendix. Exact solution on the Kagomé lattice

This method uses constructive arguments directly with generating functions in a manner similar to those arguments used in a proof via algebraic languages. For brevity I have used the generating functions directly rather than formally showing the algebraic languages upon which they are based. Herein, a generating function

$$G(x) = \sum_{n=0}^{\infty} g_n x^n \tag{A1}$$

describes the number of objects g_n with n sites, not including the base site which will be specifically mentioned in the definition of G. Note that *all* the variables mentioned hereafter will be functions of x unless specified otherwise, so the x subscript shall be dropped.

The problem is to count the number of directed site animals on the Kagomé lattice, disallowing loops. Call this generating function B for base. Let it have a base site of type 0 (see any of the pictures drawn of the Kagomé lattice with numbers inside sites indicating their 'type'). Call the generating function starting from a type-1 site A. By symmetry, the generating function from a type-2 site will also be A. Then this is a one-dimensional problem: $A = 1 + xA + x + x^2A$ as before, so

$$A = \frac{1+x}{1-x-x^2}\,. \tag{A2}$$

Define the generating functions $C_{ab}(n)$ to represent animals with a base consisting of two sites, numbered a and b in figure 1, in the diagonal perpendicular to the preferred direction. n is the distance apart of the the two sites, with 0 being as close as possible, one being the next to closest pair, etc. a is the lower right site.

By symmetry, $C_{11}(n) = C_{22}(n)$. Also note that $B = C_{21}(0)$. We can then get a set of equations governing growth:

$$C_{12}(n) = (1+x)^2(1+2xA) + x^2C_{21}(n) + x^3C_{11}(n) + x^3C_{22}(n) + x^4C_{12}(n+1) \qquad n \geqslant 0 \tag{A3}$$

$$C_{11}(n) = (1+x)^2(1+2xA) + x^2C_{22}(n) + x^3C_{21}(n) + x^3C_{12}(n+1) + x^4C_{11}(n) \qquad n \geqslant 0 \tag{A4}$$

$$C_{21}(n) = (1+x)^2(1+2xA) + x^2C_{12}(n) + x^3C_{11}(n-1) + x^3C_{22}(n-1) + x^4C_{21}(n-1) \qquad n \geqslant 1 \tag{A5}$$

$$C_{21}(0) = 1 + 2xA + x^2C_{12}(0)\,. \tag{A6}$$

To see where (A3)–(A5) come from, take for instance the C_{12} case. The 1-site can go to no sites, a 2-site, an 0-site, or the 0- and 1-sites. The 2-site has a similar set of four choices. The 16 combinations are shown in table A1.

Table A1. The 16 possible results of extending C_{12} to the next diagonal. The column represents where the right site goes to, the row represents what happens to the left site.

Go to	—	2	0	01
—	1	xA	x	x^2A
1	xA	x^2C_{21}	x^2A	x^3C_{11}
0	x	x^2A	x^2	x^3A
02	x^2A	x^3C_{22}	x^3A	x^4C_{12}

Equations (A3) and (A4) can be simplified by defining $\phi = (1+x)^2(1+2xA)$, $E(n) = C_{12}(n)$, $F(n) = C_{21}(n)$, $\alpha = 2x^3(1-x^2-x^4)^{-1}$, $\Phi = \phi(1+\alpha)$, $\beta = x^2 + \alpha x^3$, $\gamma = x^4 + \alpha x^3$ and $\theta = 1 + 2xA$. Then with a little algebra, $C_{11}(n) = C_{22}(n)$ can be eliminated, leaving

$$E(n) = \Phi + \beta F(n) + \gamma E(n+1) \qquad n \geqslant 0 \tag{A7}$$

$$F(n) = \Phi + \beta E(n) + \gamma F(n-1) \qquad n \geqslant 1 \tag{A8}$$

$$F(0) = \theta + x^2 E(0). \tag{A9}$$

Note that with the boundary conditions that no negative powers of x are allowed to appear in either $E(n)$ or $F(n)$, this is sufficient information to calculate any of the above generating functions to an arbitrary power of x, by noticing that all recursion (on the right-hand side) has a factor in front of it that is at least $O(x^2)$. This boundary condition will be used later in the actual solution.

Alternatively, given $F(0)$ and $E(0)$, one could use the above equations to solve for all $F(n)$ and $E(n)$. Define $\Psi = \Phi/(1-\beta-\gamma)$, and define $F(n) = \Psi + f(n)$, and $E(n) = \Psi + e(n)$. Then a little algebra transforms (A7) and (A8) into the homogenous pair

$$e(n) = \beta f(n) + \gamma e(n+1) \qquad n \geqslant 0 \tag{A10}$$

$$f(n) = \beta e(n) + \gamma f(n-1) \qquad n \geqslant 1. \tag{A11}$$

These are now two homogenous equations, solvable by a linear combination of linearly independent solutions. There should be exactly two solutions, as there are exactly two free variables ($F(0)$ and $E(0)$) in the inhomogenous case (equations (A7) and (A8)).

Trying a solution of the form $e(n) = \lambda^n$ and $f(n) = Ce(n)$, substituted into (A10) and (A11) gives

$$\gamma\lambda^2 - (1-\beta^2+\gamma^2)\lambda + \gamma = 0. \tag{A12}$$

This has two solutions, which means that all the possible solutions are of this form. One of these solutions has a term $O(\gamma^{-1})$, and is thus disallowed by the boundary condition that $E(n)$ and $F(n)$ have no negative powers of x. Thus there is only one solution, given by

$$\lambda = \frac{2\gamma}{\rho + \sqrt{\rho^2 - 4\gamma^2}} \tag{A13}$$

$$C = \frac{1-\gamma\lambda}{\beta} \tag{A14}$$

where $\rho = 1 - \beta^2 + \gamma^2$.

Then we can use this solution to get the solution to (A7) and (A8)

$$E(n) = \Psi + M\lambda^n \qquad n \geqslant 0 \tag{A15}$$

$$F(n) = \Psi + MC\lambda^n \qquad n \geqslant 0. \tag{A16}$$

Using the second boundary condition (equation (A9)) we can solve for M:

$$M = \frac{\theta - \Psi + x^2\Psi}{C - x^2}. \tag{A17}$$

This means we can get a final expression for B, the generating function desired in the first place as

$$B = F(0) = \Psi + MC. \tag{A18}$$

Note that in the limit $n \to \infty$, $E(\infty) = F(\infty) = \Psi = A^2$, which is as expected as it represents two independent single directed walks. This provides a check on the algebra.

This was checked by performing a series expansion of the above result, giving a result that agreed with the geometrical enumeration

$$B = 1 + 2x + 5x^2 + 10x^3 + 20x^4 + 36x^5 + 66x^6 + 118x^7 + 210x^8 + 364x^9 + 633x^{10} \\ +1\,088x^{11} + 1\,869x^{12} + 3\,176x^{13} + 5\,402x^{14} + 9\,126x^{15} + 15\,419x^{16} \\ +25\,900x^{17} + 43\,523x^{18} + 72\,822x^{19} + 121\,868x^{20} + 203\,204x^{21} \\ +338\,905x^{22} + 563\,568x^{23} + 937\,321x^{24} + 1\,555\,042x^{25} + 2\,580\,287x^{26} \\ +4\,272\,438x^{27} + 7\,075\,274x^{28} + 11\,695\,436x^{29} + 19\,335\,016x^{30} \\ +31\,914\,198x^{31} + 52\,683\,124x^{32} + 86\,846\,826x^{33} + 143\,179\,288x^{34} \\ +235\,761\,004x^{35} + 388\,242\,544x^{36} + 638\,643\,148x^{37} + O(x)^{38}.$$

An analysis of this around the critical point $(\sqrt{5} - 1)/2$ gives an expansion of $C(x - x_c)^{-1} + O(1)$, which indicates a growth like $n^0\mu^n$. A closed form expansion for the generating function B or even for the constant C is exceedingly messy and is thus not included here, though it can be easily obtained through the previous argument.

References

[1] Conway A R, Brak R and Guttmann A J 1993 Directed animals on two dimensional lattices *J. Phys. A: Math. Gen.* **26** 3085–91

[2] Dhar D, Phani M H and Barma M 1982 Enumeration of directed site animals on two-dimensional lattices *J. Phys. A: Math. Gen.* **15** L279–84

[3] Dhar D 1982 Equivalence of the two dimensional directed animal problem to Baxter hard-square lattice-gas model *Phys. Rev. Lett.* **59** 853–6

[4] Delest M 1991 Polyominos and animals: some recent results *J. Math. Chem.* **8** 3–18

[5] Duarte J A M S 1990 Directed lattices: site-to-bond conversion and its uses for the percolation and dilute polymer transitions *Z. Phys.* B **80** 299–304

[6] Guttmann A J 1989 *Asymptotic Analysis of Coefficients* ed C Domb and J Lebowitz *Phase Transitions and Critical Phenomena* vol 13 (London: Academic)